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RADIATIVE TRANSFER IN A CONSERVATIVE FINITE SLAB WITH AN INTERNAL SOURCE

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Abstract—The normal-mode expansion technique is used to obtain the radiation intensity in a conservative finite medium with an internal source and plane-parallel emitting boundaries. The elementary solutions of the one-dimensional equation of transfer and existing completeness and orthogonality theorems are used to effect the desired solution with a minimum of manipulation. The unknown expansion coefficients appearing in the solution are shown to satisfy simple integral equations to which highly accurate analytical approximations are obtained. Further, a high-order improved Gaussian quadrature integration procedure is used to construct numerically the required expansion coefficients, and thus the two universal functions $\Theta(\tau)$ and $\Theta_s(\tau)$ are calculated to "bench mark" accuracy. Since the radiation intensity is determined explicitly, all other quantities of interest, such as the incident radiation, the heat flux and the temperature distribution, are immediately available.

I. INTRODUCTION

THE PURPOSE of this paper is to illustrate the advantage to which Case's normal-mode expansion technique [1] may be used to solve a certain class of radiative heat transfer problems in finite plane-parallel media. In particular, a procedure alternative to that used by Heaslet and Warming [2] for a problem involving radiative transport and wall temperature slip in a finite, absorbing, emitting gray medium is discussed, and an explicit result for the radiation intensity in a finite conservative medium with an internal source is presented.

In one of the earlier papers to make use of the singular eigenfunction expansion technique for heat transfer applications, Ferziger and Simmons [3] considered the source-free, finite-slab problem for a conservative medium with emitting and reflecting boundaries. In addition to making use of the merits of the Case technique, Ferziger and Simmons illustrated the computational advantages of their work, and established the validity of their highly accurate analytical approximations.

An exhaustive study of radiative heat transfer problems in non-conservative media has been made by Heaslet and Warming [4] who, in addition to making use of the method of normal modes, discuss many of the interrelationships between Case's method and several other techniques. Although Heaslet and Warming emphasized isotropic coherent scattering, it is clear that similar analysis may be used to advantage when more general scattering laws are admitted.

More recently, Özişik and Siewert [5] have employed the singular eigenfunction method to solve for the radiation intensity in an absorbing, emitting, and scattering medium confined between reflecting and emitting plates. In that paper semi-analytical solutions, analogous to those found to be highly accurate by Ferziger and Simmons [3], were obtained for various inhomogeneous source terms, and the finite-medium Green's function was discussed.

There is of course a great deal of literature on the subject of radiative heat transfer in participating media; for the sake of brevity here, the reader is referred to the paper by Heaslet and Warming [2] where an extensive bibliography is given.

II. FORMULATION OF BASIC EQUATIONS

We consider the steady-state one-dimensional equation of transfer

$$\mu \frac{\partial}{\partial \tau} I(\tau, \mu) + I(\tau, \mu) = \frac{1}{2} \int_{-1}^{1} I(\tau, \mu') \, d\mu' + Q(\tau), \tag{1}$$

where $I(\tau, \mu)$ is the radiation intensity, μ is the direction cosine (as measured from the *positive* τ axis) of the propagating radiation, τ is the optical variable, and $Q(\tau)$ is an inhomogeneous source term. For finite-media problems with prescribed boundary conditions, a solution to equation (1) is sought, subject to

$$I(0, \mu) = f_1(\mu), \mu \in (0, 1), \tag{2a}$$

and

$$I(\tau_0, -\mu) = f_2(\mu), \mu \in (0, 1). \tag{2b}$$

Although it need not be the case, we take $f_1(\mu)$ and $f_2(\mu)$ to be given functions specifying the conditions on $I(\tau, \mu)$ at the two surfaces $\tau = 0$ and $\tau = \tau_0$; Ferziger and Simmons [3] and Özişik and Siewert [5] have discussed the manner in which the case of reflecting boundaries may be analyzed in light of the Case-method.

As an alternative statement of the given problem, equation (1) with equations (2) may be integrated to yield an integral equation for the radiation density (incident radiation)

$$\rho(\tau) \triangleq \int_{-1}^{1} I(\tau, \mu') \, d\mu'; \tag{3}$$

it follows that $\rho(\tau)$ is a solution of

$$\rho(\tau) = \int_0^1 f_1(\mu) \, e^{-\tau/\mu} \, d\mu + \int_0^1 f_2(\mu) \, e^{-(\tau_0 - \tau)/\mu} \, d\mu + \int_0^{\tau_0} Q(\tau') \, E_1(|\tau - \tau'|) \, d\tau' + \tfrac{1}{2} \int_0^{\tau_0} \rho(\tau') \, E_1(|\tau - \tau'|) \, d\tau', \tag{4}$$

where $E_1(x)$ is the first-order exponential integral:

$$E_N(x) = \int_0^1 \mu^{N-2} \, e^{-x/\mu} \, d\mu.$$

If consideration is restricted to the case where $f_1(\mu)$, $f_2(\mu)$ and $Q(\tau)$ are constants f_1, f_2 and Q, then equation (4) reduces to the simpler form

$$\rho(\tau) = f_1 E_2(\tau) + f_2 E_2(\tau_0 - \tau) + Q[2 - E_2(\tau) - E_2(\tau_0 - \tau)] + \tfrac{1}{2} \int_0^{\tau_0} \rho(\tau') \, E_1(|\tau - \tau'|) \, d\tau'. \tag{5}$$

In their work on radiative transport and wall temperature slip, Heaslet and Warming [2] express the temperature distribution and subsequent quantities of interest in terms of two universal functions $\Theta(\tau)$ and $\Theta_s(\tau)$ which are solutions to the equations

$$\Theta(\tau) = \tfrac{1}{2} E_2(\tau) + \tfrac{1}{2} \int_0^{\tau_0} \Theta(\tau') \, E_1(|\tau - \tau'|) \, d\tau' \tag{6a}$$

and

$$\Theta_s(\tau) = \tfrac{1}{4} + \tfrac{1}{2} \int_0^{\tau_0} \Theta_s(\tau') \, E_1(|\tau - \tau'|) \, d\tau'. \tag{6b}$$

It is now clear that $\Theta(\tau)$ corresponds to $\rho_1(\tau)$ for the case $Q = f_2 = 0$ and $f_1 = \frac{1}{2}$; similarly $\Theta_s(\tau)$ is equivalent to the solution $\rho_2(\tau)$ corresponding to the case $Q = f_1 = f_2 = \frac{1}{8}$.

Ferziger and Simmons [3] have used the Case-method to solve the source-free problem above, and have demonstrated the computational merits of their semi-analytical solution. In the next section, similar analysis is used to develop a solution which includes both cases.

III. GENERAL ANALYSIS

We seek a solution to equation (1) subject to the boundary conditions given by equations (2); we restrict our attention to the case where Q, f_1 and f_2 are constants, and thus we shall obtain solutions to equation (5) from the more general result for the radiation intensity $I(\tau, \mu)$.

Since the normal modes of the homogeneous equation of transfer are established [1, 6], the desired solution can be written as

$$I(\tau, \mu) = A_+ \tfrac{1}{2} + A_- \tfrac{1}{2}(\tau - \mu) + \int_{-1}^{1} A(\eta) \, \phi(\eta, \mu) \, e^{-\tau/\eta} \, d\eta + I_p(\tau, \mu), \tag{7}$$

where A_+, A_- and $A(\eta)$ are the unknown expansion coefficients to be determined from equations (2). In addition $I_p(\tau, \mu)$ denotes a particular solution to the inhomogeneous equation of transfer. Lundquist and Horak [7] have compiled a very useful table of particular solutions; their relevant result is quoted here:

$$I_p(\tau, \mu) = Q[-\tfrac{3}{2}\tau^2 + 3\mu\tau - 3\mu^2]. \tag{8}$$

The generalized function $\phi(\eta, \mu)$ appearing in the solution takes the form [1]

$$\phi(\eta, \mu) = \frac{\eta}{2}\frac{P}{\eta - \mu} + [1 - \eta \tanh^{-1}\eta]\,\delta(\eta - \mu), \tag{9}$$

where the symbol P is used to indicate that all ensuing integrals over μ or η are to be evaluated in the Cauchy principal-value sense, and $\delta(x)$ denotes the Dirac delta function.

Equation (7) may be integrated immediately to yield results for the radiation density and the net radiative heat flux,

$$q(\tau) \triangleq \int_{-1}^{1} I(\tau, \mu')\,\mu'\,d\mu'. \tag{10}$$

It follows that

$$\rho(\tau) = A_+ + A_-\tau + \int_{-1}^{1} A(\eta)\,e^{-\tau/\eta}\,d\eta - Q(3\tau^2 + 2), \tag{11}$$

and

$$q(\tau) = -\tfrac{1}{3}A_- + 2Q\tau. \tag{12}$$

If for the considered problem the radiative transfer mechanism is interpreted as an absorbing and emitting phenomenon, rather than as a scattering process, then the temperature distribution is also at once available since, as shown by Heaslet and Warming [2], it is related in a simple manner to $\rho(\tau)$.

It is noted that the solution given by equation (7) rigorously satisfies the considered equation of transfer, and thus that the essence of the Case-method is concerned with the determination of the unknown expansion coefficients A_+, A_- and $A(\eta)$. These coefficients must, of course, be constructed such that $I(\tau, \mu)$ meets the boundary conditions of the problem; however, once these coefficients are obtained all other quantities of interest follow immediately. Clearly for the case $Q = f_1 = f_2 = \frac{1}{8}$, equation (12) yields an exact result for the heat flux, since the symmetry of the problem requires that $q(\tau_0/2) = 0$, and thus $A_- = \frac{3}{8}\tau_0$.

We proceed by substituting equation (7) into equation (2) and arranging the terms to read

$$f_1 + 3\mu^2 Q + \tfrac{1}{2}\mu A_- - \int_0^1 A(-\eta)\,\phi(-\eta, \mu)\,d\eta = \tfrac{1}{2}A_+ + \int_0^1 A(\eta)\,\phi(\eta, \mu)\,d\eta,\ \mu \in (0, 1), \tag{13a}$$

and

$$f_2 + (\tfrac{3}{2}\tau_0^2 + 3\mu\tau_0 + 3\mu^2)\,Q - \tfrac{1}{2}(\tau_0 + \mu)\,A_- - \int_0^1 A(\eta)\,\phi(-\eta, \mu)\,e^{-\tau_0/\eta}\,d\eta = \tfrac{1}{2}A_+ + \int_0^1 A(-\eta)\,\phi(\eta, \mu)\,e^{\tau_0/\eta}\,d\eta,\ \mu \in (0, 1). \tag{13b}$$

Equations (13) must yield the desired solutions for A_+, A_- and $A(\eta)$. These equations are singular; however, they may be converted simply to Fredholm-type equations by utilizing the existing half-range orthogonality relations given by Kuščer *et al.* [8]. We prefer not to use the standard X-function notation [8], but rather to make use of the half-range weight function $\mu H(\mu)$, where $H(\mu)$ is Chandrasekhar's H-function [9] corresponding to characteristic function $\Psi(\mu) = \frac{1}{2}$.

Equations (13) are thus multiplied by $\mu H(\mu)$ and integrated over μ from zero to unity; the resulting two equations may be written in the convenient matrix form:

$$\mathbf{MA} = \mathbf{G} + \int_0^1 \mathbf{B}(\eta')\,\mathbf{A}(\eta')\,\frac{1}{H(\eta')}\,\eta'\,d\eta', \tag{14}$$

where the unknowns are now expressed as

$$\mathbf{A} = \begin{vmatrix} A_+ \\ A_- \end{vmatrix} \quad \text{and} \quad \mathbf{A}(\eta) = \begin{vmatrix} A(\eta) \\ A(-\eta) \end{vmatrix},\quad \eta \in (0, 1). \tag{15}$$

In addition, the matrices

$$\mathbf{B}(\eta) \triangleq \begin{vmatrix} 0 & -1 \\ -e^{-\tau_0/\eta} & 0 \end{vmatrix} \text{ and } \mathbf{M} \triangleq \frac{\sqrt{3}}{3}\begin{vmatrix} 1 & -z_0 \\ 1 & \tau_0 + z_0 \end{vmatrix} \quad (16)$$

have been defined, and

$$\mathbf{G} = \tfrac{2}{3}(\sqrt{3})\begin{vmatrix} f_1 + 3Qz_1 \\ f_2 + 3Q(\frac{1}{2}\tau_0^2 + \tau_0 z_0 + z_1) \end{vmatrix}, \quad (17)$$

where

$$z_0 = \tfrac{1}{2}(\sqrt{3})\int_0^1 H(\mu)\,\mu^2\,d\mu = 0{\cdot}71044609, \quad (18a)$$

and

$$z_1 = \tfrac{1}{2}(\sqrt{3})\int_0^1 H(\mu)\,\mu^3\,d\mu = 0{\cdot}55236682. \quad (18b)$$

The continuum coefficients on the right-hand sides of equations (13) are isolated similarly by multiplying those equations by $\mu H(\mu)\,\phi(\eta', \mu)$, $\eta' \in (0, 1)$, and integrating over $\mu \in (0, 1)$. These results also can be written more conveniently in matrix notation:

$$\mathbf{M}(\eta)\,\mathbf{A}(\eta) = \mathbf{G}(\eta) + \mathbf{BA}\tfrac{1}{6}(\sqrt{3})\frac{1}{H(\eta)}g(1, \eta) + \int_0^1 \mathbf{B}(\eta')\,\mathbf{A}(\eta')\,K(\eta' \to \eta)\,d\eta', \eta \in (0, 1), \quad (19)$$

where

$$\mathbf{B} = \begin{vmatrix} 0 & -1 \\ 0 & 1 \end{vmatrix}, \quad \mathbf{M}(\eta) = \begin{vmatrix} 1 & 0 \\ 0 & e^{\tau_0/\eta} \end{vmatrix}, \quad (20)$$

and

$$\mathbf{G}(\eta) = -\frac{(\sqrt{3})\,Q}{H(\eta)}g(1, \eta)\begin{vmatrix} \eta + z_0 \\ \eta + z_0 + \tau_0 \end{vmatrix}. \quad (21)$$

In addition, the kernel is given by

$$K(\eta' \to \eta) = \tfrac{1}{2}\eta' g(1, \eta)\frac{1}{\eta + \eta'}\frac{1}{H(\eta)\,H(\eta')}, \quad (22)$$

where

$$g(1, \eta) = \left\{(1 - \eta \tanh^{-1}\eta)^2 + \frac{\pi^2\eta^2}{4}\right\}^{-1}. \quad (23)$$

Since it is highly unlikely that analytical solutions to the coupled equations (14) and (19) exist, it follows that the degree of precision with which the desired solution can be completed is measured by how accurately the expansion coefficients can be computed from equations (14) and (19). Although these equations are formidable analytically, they certainly pose no problem for existing computing facilities. Thus if highly accurate "bench mark" solutions are sought, an iterative procedure could be used to construct results valid to any reasonable degree of accuracy. Bond and Siewert [11] have solved similar equations numerically for a problem in neutron transport theory; their work illustrates the computational merits of the singular eigenfunction expansion technique.

Fortunately analytical approximations can be obtained from equations (14) and (19) which should yield solutions of sufficient accuracy. Ferziger and Simmons [3] obtained two approximate solutions to these equations for $f_2 = Q = 0$; they showed that the lowest-order solution was better than classical diffusion theory, whereas the second-order solution was nearly exact.

In the present analysis, the lowest-order solution is obtained by neglecting the continuum coefficients entirely; the discrete solutions thus are readily available from equation (14):

$$\mathbf{A}_1(\eta) \equiv 0; \qquad \mathbf{A}_1 = \mathbf{M}^{-1}\mathbf{G}. \quad (24)$$

The second-order result for the continuum solution is found by neglecting the contribution from the kernel $K(\eta' \to \eta)$ in equation (19), and by using $\mathbf{A}_1$ in that equation. Finally $\mathbf{A}_2(\eta)$ is substituted into equation (14) to yield $\mathbf{A}_2$. It follows that

$$\mathbf{A}_2(\eta) = \mathbf{M}^{-1}(\eta)\left[\mathbf{G}(\eta) + \mathbf{B}\mathbf{M}^{-1}\,\mathbf{G}\frac{\sqrt{3}}{6}\frac{1}{H(\eta)}g(1,\eta)\right] \tag{25a}$$

and

$$\mathbf{A}_2 = \mathbf{M}^{-1}\left[\mathbf{G} + \int_0^1 \mathbf{B}(\eta')\,\mathbf{A}_2(\eta')\frac{1}{H(\eta')}\eta'\,\mathrm{d}\eta'\right]. \tag{25b}$$

To summarize, the explicit results for the expansion coefficients given by equations (25) are to be used with equations (7) and (8) to give the desired solution for the radiation intensity. Since $I(\tau,\mu)$ is thus established, the other quantities of interest follow immediately, as illustrated by equations (11) and (12).

It should be noted that the present analysis may be generalized to include a linear (or higher-order polynomial) inhomogeneous source term by making use of the particular solutions given by Lundquist and Horak [7]. In fact, the only difference in the computation of the expansion coefficients will be that the vectors $\mathbf{G}$ and $\mathbf{G}(\eta)$ appearing in equations (14) and (19) will take slightly more general forms.

IV. NUMERICAL ANALYSIS

Since the analytical advantages of the singular eigenfunction expansion technique have been exhibited, we should now like to illustrate the method by constructing numerical solutions for the required expansion coefficients $\mathbf{A}$ and $\mathbf{A}(\eta)$, $\eta \in (0, 1)$. Once these expansion coefficients are established, numerical results for the universal functions [2] $\Theta(\tau)$ and $\Theta_s(\tau)$ are immediately available through the use of equation (11).

As discussed previously, we must solve the two equations

$$\mathbf{A} = \mathbf{M}^{-1}\,\mathbf{G} + \mathbf{M}^{-1}\int_0^1 \mathbf{B}(\eta')\,\mathbf{A}(\eta')\frac{1}{H(\eta')}\eta'\,\mathrm{d}\eta' \tag{26a}$$

and

$$\mathbf{A}(\eta) = \mathbf{M}^{-1}(\eta)\,\mathbf{G}(\eta) + \mathbf{M}^{-1}(\eta)\,\mathbf{B}\mathbf{A}\frac{\sqrt{3}}{6}\frac{1}{H(\eta)}g(1,\eta) + \mathbf{M}^{-1}(\eta)\int_0^1 \mathbf{B}(\eta')\,\mathbf{A}(\eta')\,K(\eta' \to \eta)\,\mathrm{d}\eta', \quad \eta \in (0, 1). \tag{26b}$$

Using an improved Gaussian quadrature scheme [12] to evaluate the integral terms in equations (26), we have solved iteratively the above equations to yield numerical results for the expansion coefficients $\mathbf{A}$ and $\mathbf{A}(\eta)$, $\eta \in (0, 1)$. The tractable analytical approximations given by equations (25) were used to initiate the calculation, and the iteration procedure was terminated when the values of $\mathbf{A}$ and $\mathbf{A}(\eta)$ evaluated at the nodal points differed after successive iterations by less than $\epsilon = 10^{-15}$

All computations here and in the determination of $\Theta(\tau)$ and $\Theta_s(\tau)$ were performed in double-precision arithmetic on an IBM 360/75 computer, and for all cases two integration procedures were used: an 81-point improved Gaussian quadrature scheme was used over the total interval $\eta \in (0, 1)$; further, this interval was divided equally and the same 81-point integration method was used in each subinterval.

Since the analytical approximations given by equations (25) proved to be highly accurate, we have given computational priority to the two special cases, $Q = f_2 = 0$ and $f_1 = \frac{1}{2}$, and $Q = f_1 = f_2 = \frac{1}{8}$, necessary to establish the two universal functions $\Theta(\tau)$ and $\Theta_s(\tau)$, discussed by Heaslet and Warming [2, 13]. We consider these calculations to be highly accurate, and thus would like to mention the further checks

Table 1. $\Theta(\tau)$ and $\Theta_s(\tau)$ for slab of optical thickness $\tau_0 = 0{\cdot}2$

τ/τ_0	$\Theta(\tau)$		$\Theta_s(\tau)$	
	Analytical approximation	Exact	Analytical approximation	Exact
0	0·6081	0·611433	0·3167	0·321694
0·05	0·5940	0·596683	0·3234	0·328063
0·10	0·5821	0·584385	0·3278	0·332147
0·15	0·5710	0·572904	0·3311	0·335313
0·20	0·5603	0·561901	0·3337	0·337843
0·25	0·5499	0·551215	0·3359	0·339866
0·30	0·5397	0·540750	0·3375	0·341455
0·35	0·5297	0·530441	0·3388	0·342655
0·40	0·5197	0·520239	0·3396	0·343495
0·45	0·5099	0·510103	0·3402	0·343992
0·50	0·5000	0·500000	0·3403	0·344157

Table 2. $\Theta(\tau)$ and $\Theta_s(\tau)$ for slab of optical thickness $\tau_0 = 1{\cdot}0$

τ/τ_0	$\Theta(\tau)$		$\Theta_s(\tau)$	
	Analytical approximation	Exact	Analytical approximation	Exact
0	0·7576	0·758146	0·5157	0·516842
0·05	0·7226	0·722979	0·5666	0·567455
0·10	0·6943	0·694563	0·6000	0·600637
0·15	0·6679	0·668163	0·6262	0·626803
0·20	0·6427	0·642872	0·6475	0·647999
0·25	0·6181	0·618285	0·6647	0·665137
0·30	0·5941	0·594170	0·6783	0·678718
0·35	0·5703	0·570381	0·6886	0·689045
0·40	0·5468	0·546809	0·6959	0·696308
0·45	0·5233	0·523372	0·7003	0·700624
0·50	0·5000	0·500000	0·7017	0·702056

Table 3. $\Theta(\tau)$ and $\Theta_s(\tau)$ for slab of optical thickness $\tau_0 = 2{\cdot}0$

τ/τ_0	$\Theta(\tau)$		$\Theta_s(\tau)$	
	Analytical approximation	Exact	Analytical approximation	Exact
0	0·8307	0·830791	0·7384	0·738729
0·05	0·7866	0·786605	0·8815	0·881646
0·10	0·7508	0·750879	0·9774	0·977575
0·15	0·7174	0·717420	1·0545	1·054567
0·20	0·6851	0·685130	1·1177	1·117785
0·25	0·6535	0·653546	1·1694	1·169444
0·30	0·6224	0·622417	1·2107	1·210721
0·35	0·5916	0·591591	1·2423	1·242305
0·40	0·5610	0·560961	1·2646	1·264618
0·45	0·5305	0·530452	1·2779	1·277915
0·50	0·5000	0·500000	1·2823	1·282332

employed to substantiate confidence in our results.

Since equation (7) is obviously a solution of the considered equation of transfer, the only point in question is how accurately that solution can be constrained to meet the boundary conditions of the problem. Thus, after constructing numerically the expansion coefficients **A** and **A**(η), ideally we should reconfirm pointwise, $\mu \in (0, 1)$, the boundary conditions given by equations (13). Since this procedure would necessitate the numerical evaluation of principal-value integrals, thus introducing further errors, we prefer to evaluate instead moments of equations (13).

In order to develop this measure of the accuracy of our calculations, we multiply equations (13) by $\mu^{\alpha}H(\mu)/f_1$ and integrate over μ from zero to unity; we find

$$H_\alpha + \frac{3}{f_1} Q H_{\alpha+2} - \frac{1}{2f_1} A_+ H_\alpha + \frac{1}{2f_1} A_- H_{\alpha+1} = - \frac{1}{2f_1} \sum_{\beta=1}^{\alpha-1} H_{\alpha-\beta} \int_0^1 A(\eta)\, \eta^\beta \, \mathrm{d}\eta + \frac{1}{2f_1} \int_0^1 A(-\eta)\, \eta \int_0^1 \mu^\alpha H(\mu) \frac{\mathrm{d}\mu}{\mu + \eta}\, \mathrm{d}\eta \qquad (27a)$$

and

$$\frac{f_2}{f_1} H_\alpha + \frac{Q}{f_1} [\tfrac{3}{2}\tau_0^2 H_\alpha + 3\tau_0 H_{\alpha+1} + 3H_{\alpha+2}] - \frac{1}{2f_1} A_+ H_\alpha - \frac{1}{2f_1} A_- [\tau_0 H_\alpha + H_{\alpha+1}] = - \frac{1}{2f_1} \sum_{\beta=1}^{\alpha-1} H_{\alpha-\beta} \int_0^1 A(-\eta)\, e^{\tau_0/\eta} \eta^\beta \, \mathrm{d}\eta + \frac{1}{2f_1} \int_0^1 A(\eta)\, \eta\, e^{-\tau_0/\eta} \int_0^1 \mu^\alpha H(\mu) \frac{\mathrm{d}\mu}{\mu + \eta}\, \mathrm{d}\eta, \qquad (27b)$$

where the moments of Chandrasekhar's H-function are [9]

$$H_\alpha \triangleq \int_0^1 \mu^\alpha H(\mu)\, \mathrm{d}\mu. \qquad (28)$$

Defining $K_1(\alpha)$ to be the relative difference between the two sides of equation (27a) and $K_2(\alpha)$ similarly with respect to equation (27b), we note that for the worst case reported here $K_1(\alpha) < K_2(\alpha) < 6 \times 10^{-7}$, where $\alpha = q, 2, 3, \ldots 8$.

Since **A** and **A**(η), $\eta \in (0, 1)$, have been established accurately, the computation of $\Theta(\tau)$ and $\Theta_s(\tau)$ follows directly from equation (11). The results of these "exact" calculations and the predictions resulting from the analytical approximation given by equations (25) are given in the accompanying tables, where for display purpose the even or odd character of the universal functions [13] has been utilized.

As a final indication of the accuracy of our "exact" calculations, the quantity

$$F(\tau_0) \triangleq \frac{4}{\tau_0} \int_0^{\tau_0} \Theta_s(\tau)\, E_2(\tau)\, \mathrm{d}\tau \qquad (29)$$

has been shown to differ from the rigorous value of unity [13] by less than 1×10^{-7}.

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TRANSPORT PAR RAYONNEMENT DANS UNE PLAQUE FINIE SANS DISSIPATION AVEC UNE SOURCE INTERNE

Résumé—La technique de développement en modes normaux est employée pour l'intensité de rayonnement dans un milieu fini non-dissipatif avec une source interne et des frontières émettrices planes et parallèles. Les solutions élémentaires de l'équation de transport unidimensionnelle et les théorèmes disponibles pour le caractère complet et l'orthogonalitè sont employés pour obtenir la solution désirée avec un minimum de manipulation. Les coefficients inconnus du développement apparaissent dans la solution satisfont à une intégrale simple pour laquelle des approximations analytiques de précision élevée sont obtenus. De plus, un processus d'intégration par une quadrature Gaussienne améliorée d'ordre élevé est employé pour construire numériquement les coefficients demandés du développement, et ainsi les deux fonctions universelles, $\theta(\tau)$ et $\theta_s(\tau)$ sont calculées avec une précision "d'atelier". Puisque l'intensité du rayonnement est déterminée explicitement, toutes les autres quantités intéressantes, telles que le rayonnement incident, le flux de chaleur et la distribution de température sont immédiatement disponibles.

STRAHLUNGSAUSTAUSCH IN EINEM GEWÖHNLICHEN, ENDLICHEN SPALT MIT EINER INNEREN QUELLE.

Zusammenfassung—Die "normal-mode expansion technique" wird verwendet um Lösungen für die Strahlungsintensität in einem gewöhnlichen, endlichen Medium mit einer inneren Quelle und planparallelen, emittierenden Grenzflächen zu finden. Die elementaren Lösungen der eindimensionalen Transportgleichung und bekannte Vollständigkeits- und Orthogonaltheoreme werden verwendet um die gesuchte Lösung mit einem Minimum an Aufwand zu erhalten. Es wird gezeigt, dass die unbekannten Entwicklungskoeffizienten der Lösung einfachen Integralgleichungen genügen, für die sich sehr genaue analytische Näherungen finden lassen. Ferner wird eine verbesserte Gauss'sche Integrationsprozedur höherer Ordnung verwendet um numerisch die erforderlichen Expansionskoeffizienten zu konstruieren; und damit die beiden universalen Funktionen $\theta(\tau)$ und $\theta_s(\tau)$ auf "bench-mark"-Genauigkeit zu berechnen. Da die Strahlungsintensität explizit bestimmt wird, kann man alle anderen interessierenden Grössen, wie die einfallende Strahlung, den Wärmestrom und die Temperaturverteilung sofort erhalten.

ЛУЧИСТЫЙ ПЕРЕНОС В КОНСЕРВАТИВНОЙ КОНЕЧНОЙ ПЛИТЕ С ВНУТРЕННИМ ИСТОЧНИКОМ

Аннотация—Для решения задачи об интенсивности излучения в консервативной конечной среде с внутренним источником и плоскопараллельными излучающими границами используется техника разложения по нормальным модям. Чтобы получить нужное решение наиболее простым способом, используются элементарные решения одномерного уравнения переноса и существующие теоремы полноты и ортогональности-Показано, что неизвестные коэффициенты разложения, появляющиеся в решении. удовлетворяют простому интегралу, для которого получены аналитические аппроксимации большой точности. Далее используя процедуру интегрирования посредством улучшения Гауссовских квадратур высокого порядка, численно находятся нужные коэффициенты разложения и, таким образом, рассчитываются две универсальные функции $\theta(\tau)$ и $\theta_s(\tau)$ с заданной наперед точностью.

Поскольку интенсивность излучения определяется в явном виде, все другие величины, представляющие интерес, такие как падающее излучение, тепловой поток и распределение температуры находятся сразу же.